

# The Riemann solution to the Chaplygin pressure Aw-Rascle model with Coulomb-like friction and its vanishing pressure limit

Qingling Zhang

*School of Mathematics and Computer Sciences, Jiangnan University, Wuhan 430056, PR China*

---

## Abstract

The Riemann solution to the Chaplygin pressure Aw-Rascle model with Coulomb-like friction is constructed explicitly and its vanishing pressure limit is analyzed precisely. It is shown that the delta shock wave appears in the Riemann solutions in some certain situations. The generalized Rankine-Hugoniot conditions of the delta shock wave are established and the exact position, propagation speed and strength of the delta shock wave are given explicitly, which enables us to see the influence of the Coulomb-like friction on the Riemann solution to the Chaplygin pressure Aw-Rascle model clearly. It is shown that the Coulomb-like friction term makes contact discontinuities and delta shock waves bend into parabolic shapes and the Riemann solutions are not self-similar anymore. Finally, the occurrence mechanism on the phenomenon of concentration and cavitation and the formation of delta shock wave and vacuum in the process of vanishing pressure limit are analyzed and identified in detail. Moreover, we show the Riemann solutions to the non-homogeneous Chaplygin pressure Aw-Rascle model converge to the Riemann solutions to the transportation equations with the same source term as the pressure vanishes. These two results generalize those obtained in [7, 38] for homogeneous equations to nonhomogeneous equations and are also applicable to the nonsymmetric system of Keyfitz-Kranzer type with the same Chaplygin pressure and Coulomb-like friction.

*Keywords:* Chaplygin pressure; Aw-Rascle model; Riemann solutions; delta shock wave; Coulomb-like friction; vanishing pressure limit.

*2010 MSC:* 35L65, 35L67, 35B30, 76N10

---

## 1. Introduction

In this paper, we are mainly concerned with the Riemann problem for the Chaplygin pressure Aw-Rascle model with Coulomb-like friction

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho(u + P))_t + (\rho u(u + P))_x = \beta \rho, \end{cases} \quad (1.1)$$

with Riemann initial data

$$(\rho, u)(x, 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \quad (1.2)$$

where  $\rho_{\pm}$  and  $u_{\pm}$  are all given constants. In (1.1), the state variable  $\rho > 0$  and  $u \geq 0$  denote the traffic density and velocity, respectively,  $\beta$  is a frictional constant, and the pressure  $P$  is given by the state equation

$$P = -\frac{A}{\rho}, \quad A > 0, \quad (1.3)$$

which was introduced by Chaplygin [6] and Tsien[43] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics.

The Euler system with state equation (1.3) is the classical Chaplygin gas equations which has been advertised as a possible model for dark energy of the universe [3, 16] and have been extensively investigated recently [5, 19, 30, 47] etc. The generalized Chaplygin gas model has also attracted intensive attention such as in [3, 31, 38, 44]. It can be used to describe the dark matter and dark energy in the unified form through exotic background fluid whose state equation is given by  $P = -\frac{A}{\rho^\alpha}$ ,  $0 < \alpha < 1$ ,  $A > 0$ . The modified Chalygin gas was proposed by Benaoum in 2002 [2] to describe the current accelerated expansion of the universe, whose equation of state is given by  $P = A\rho - \frac{B}{\rho^\alpha}$ ,  $0 < \alpha \leq 1$ ,  $A, B > 0$ . Compared with the Chalygin gas or the generalized Chalygin gas, the model for the modified Chalygin gas can describe the universe to a large extent.

If  $\beta = 0$ , then the system (1.1) becomes the Chaplygin pressure Aw-Rascle model which was recently introduced by Pan and Han [28], in which delta-shocks appear in the Riemann solutions, which may be used to explain the serious traffic jam. Sheng and Zeng [39] considered its Riemann problem with delta initial data. With these results, similar to [20, 45, 46], we recently solved the Cauchy problem of it in [48] by generalized potential method. If  $\beta = 0$  and  $P = \rho^\gamma$ ,  $\gamma > 0$ , then the system (1.1) becomes the classical Aw-Rascle model of traffic flow proposed by Aw and Rascle [1] in 2000 to remedy the deficiencies of second order models of car traffic pointed out by Daganzo [11] and had also been independently derived by Zhang [49]. Since then, it had received extensive attention [17, 25, 34, 36]. Recently, the Riemann problem for the Aw-Rascle model with generalized Chaplygin pressure was also considered by Guo in [18] in which the delta-shock also appears. Cheng and Yang [10] considered the Riemann problem for the Aw-Rascle model with modified Chaplygin pressure  $P = A\rho - \frac{B}{\rho}$ ,  $A, B > 0$  and analyzed the limit of its Riemann solutions with the pressure approaching Chaplygin pressure.

In fact, if  $\beta = 0$ ,  $P = \rho^\gamma$ ,  $\gamma > 0$  and let  $u = w - P$ , then the classical Aw-Rascle model can be written as the nonsymmetric system of Keyfitz-Kranzer type as follows:

$$\begin{cases} \rho_t + (\rho(w - P))_x = 0, \\ (\rho w)_t + (\rho w(w - P))_x = 0, \end{cases} \quad (1.4)$$

Recently, Lu [27] studied the existence of global entropy solution to general system of Keyfitz-kraner type (1.4) with state equation  $P = P(\rho)$  satisfying some conditions. In 2013, Cheng [8, 9] considered the Riemann problem of (1.4) with different choice of state equation of  $P$ , such as  $P$  taken as the Chaplygin pressure, the generalized Chaplygin pressure and the modified Chaplygin pressure, etc, which showed that the Riemann solutions

to (1.4) with Chaplygin pressure and generalized Chaplygin pressure were very similar to that of the Aw-Rascle model with the corresponding pressure.

If  $\beta = 0$  and  $P = 0$ , then the system (1.1) becomes the so-called zero pressure flow (transportation equations). It is well known that the delta-shock wave also appears in the Riemann solutions to the zero pressure flow which has been widely studied such as [4, 14, 20, 26, 40, 45, 46]. Recently, Shen [32] considered (1.1) with  $P = 0$  and solved the Riemann problem and the generalized Riemann problem for the transportation equations Coulomb-like friction. Delta-shock is a very interesting topic in the theory of conservation laws. It is a generalizations of an ordinary shock. Speaking informally, it is a kind of discontinuity, on which at least one of the variables may be develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. From the physical point of view, it represents the process of the concentration of the mass. For related research of delta-shock waves, we refer readers to papers [24, 26, 40, 41, 42] and the references cited therein for more details.

From the above discussions, one can see that the Riemann problem for the Aw-Rascle model with various kinds of pressure but without source term (namely  $\beta = 0$ ) has been well investigated. Hence, it is natural to expect the study of it with a source term, such as damping, friction and relaxation effect. In the present paper, we want to deal with the Riemann problem for the Chaplygin pressure Aw-Rascle model with Coulomb-like friction which was proposed by Savage and Hutter in 1989 [29] to describe granular flow behavior. For research on other models with Coulomb-like friction, one can see [32, 33, 37].

In this paper, we are interested in how the delta-shock solution of the Chaplygin pressure Aw-Rascle model with Coulomb-like friction develops under the influence of the Coulomb-like friction. The advantage of this kind source term is in that (1.1) can be written in a conservative form such that exact solutions to the Riemann problem (1.1) and (1.2) can be constructed explicitly. We shall see that the Riemann solutions to (1.1) and (1.2) are not self-similar any more, in which the state variable  $u$  varies linearly along with the time  $t$  under the influence of the Coulomb-like friction. In other words, the state variable  $u - \beta t$  remains unchanged in the left, intermediate and right states. In some situations, the delta-shock wave appears in the Riemann solutions to (1.1) and (1.2). In order to describe the delta-shock wave, the generalized Rankine-Hugoniot conditions are derived and the exact position, propagation speed and strength of the delta shock wave are obtained completely. It is shown that the Coulomb-like friction term make contact discontinuities and delta shock waves bend into parabolic shapes for the Riemann solutions.

Finally, the occurrence mechanism on the phenomenon of concentration and cavitation and the formation of delta shock wave and vacuum in the process of vanishing pressure limit of Riemann solutions to the nonhomogeneous Chaplygin pressure Aw-Rascle model are analyzed and identified in detail, from which we find that there is something different from polytropic gas in [7] but similar to generalized Chaplygin gas in [38] about the formation of the delta shock wave. Moreover, we show the Riemann solutions to the nonhomogeneous Chaplygin pressure Aw-Rascle model converge to the Riemann solutions for the transportation equations with the same source term as the pressure vanishes. These two results generalize those obtained in [7, 38] for homogeneous equations to nonhomogeneous equations. Since the configuration of the Riemann solution to (1.4) with Chaplygin pressure is very similar to that of (1.1) with  $\beta = 0$  (see [8]), we can obtain similar re-

sults for the nonsymmetric system of Keyfitz-Kranzer type (1.4) with the same Chaplygin pressure and Coulomb-like friction.

In fact, it was shown in [18] that the delta-shock also appears in the Riemann solutions to the Aw-Rascle model with generalized Chaplygin pressure. It should be remarkable that a significant mathematical difference between the Aw-Rascle model with generalized Chaplygin pressure and with Chaplygin pressure for the reason that there is one characteristic field genuinely nonlinear for the former, whose elementary waves admit not only contact discontinuities, but also rarefaction waves and shock waves, while the two characteristic fields are all linearly degenerate for the latter, whose elementary waves admit only contact discontinuities. To investigate how the Coulomb-like friction affects the rarefaction waves, shock waves and the delta shock waves and the occurrence mechanism of the delta shock waves in the process of pressure decreasing, we will study the Riemann problem for the generalized Chaplygin pressure Aw-Rascle model with Coulomb-like friction and its vanishing pressure limit, whose results will also be applicable to the nonsymmetric system of Keyfitz-Kranzer type (1.4) with the same pressure and Coulomb-like friction.

This paper is organized as follows. In Section 2, the system (1.1) is reformulated into a conservative form and then some general properties of the conservative form are obtained. Then, the exact solution to the Riemann problem for the conservative form are constructed explicitly, which involves the delta shock wave. Furthermore, the generalized Rankine-Hugoniot conditions are established and the exact position, propagation speed and strength of the delta shock wave are given explicitly. In Section 3, the generalized Rankine-Hugoniot conditions and the exact Riemann solutions to (1.1) and (1.2) are also given. Furthermore, it is proven rigorously that the delta-shock wave is indeed a weak solution to the Riemann problem (1.1) and (1.2) in the sense of distributions. In Section 4, we analyze the formation of delta shock waves and vacuum states in the Riemann solutions to (1.1) and (1.2) in the vanishing pressure limit and show that the Riemann solutions converge to the corresponding ones of the transportation equations with the same source term as the pressure vanishes. Finally, conclusions and discussions are carried out in Section 5.

## 2. Riemann problem for a modified conservative system

In this section, we are devoted to the study of the Riemann problem for a conservative system (1.1) in detail. Let us introduce the new velocity  $v(x, t) = u(x, t) - \beta t$ , then the system (1.1) can be reformulated in a conservative form as follows:

$$\begin{cases} \rho_t + (\rho(v + \beta t))_x = 0, \\ (\rho(v + P))_t + (\rho(v + P)(v + \beta t))_x = 0. \end{cases} \quad (2.1)$$

In fact, the change of variable was introduced by Faccanoni and Mangeney [15] to study the shock and rarefaction waves of the Riemann problem for the shallow water equations with a Coulomb-like friction term. Here, we use this transformation to study the delta shock wave for the system (1.1) which is a fully linearly degenerate system.

Now we want to deal with the Riemann problem for the conservative system (2.1) with the same Riemann initial data (1.2) as follows:

$$(\rho, v)(x, 0) = \begin{cases} (\rho_-, u_-), & x < 0, \\ (\rho_+, u_+), & x > 0. \end{cases} \quad (2.2)$$

We shall see hereafter that the Riemann solutions to (1.1) and (1.2) can be obtained immediately from the Riemann solutions to (2.1) and (2.2) by using the transformation of state variables  $(\rho, u)(x, t) = (v + \beta t)(x, t)$ .

The system (2.1) can be rewritten in the quasi-linear form

$$\begin{pmatrix} 1 & 0 \\ v & \rho \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} v + \beta t & \rho \\ v(v + \beta t) & \rho(2v + \beta t + P) \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.3)$$

It can be derived directly from (2.3) that the conservative system (2.1) has two eigenvalues

$$\lambda_1(\rho, v) = v + \beta t - \frac{A}{\rho}, \quad \lambda_2(\rho, v) = v + \beta t,$$

whose corresponding right eigenvectors can be expressed respectively by

$$r_1 = (\rho, -\frac{A}{\rho})^T, \quad r_2 = (1, 0)^T.$$

So (2.1) is strictly hyperbolic for  $\rho > 0$ . Moreover,  $\nabla \lambda_i \cdot r_i = 0$ ,  $i = 1, 2$ , which implies that  $\lambda_1$  and  $\lambda_2$  are both linearly degenerate and the associated waves are both contact discontinuities denoted by  $J$ , see [35].

We should take notice the fact that the parameter  $t$  only appears in the flux functions in the conservative system (2.1), such that the Rankine-Hugoniot conditions can be derived in a standard method as in [35]. For a bounded discontinuity at  $x = x(t)$ , let us denote  $\sigma(t) = x'(t)$ , then the Rankine-Hugoniot conditions for the conservative system (2.1) can be expressed as

$$\begin{cases} -\sigma(t)\rho + [\rho(v + \beta t)] = 0, \\ -\sigma(t)[\rho(v + P)] + [\rho(v + P)(v + \beta t)] = 0, \end{cases} \quad (2.4)$$

where  $[\rho] = \rho_r - \rho_l$  with  $\rho_l = \rho(x(t) - 0, t)$ ,  $\rho_r = \rho(x(t) + 0, t)$ , in which  $[\rho]$  denote the jump of  $\rho$  across the discontinuity, etc. It is clear that the propagation speed of the discontinuity depends on the parameter  $t$ , which is obviously different from classical hyperbolic conservation laws.

If  $\sigma(t) \neq 0$ , then it follows from (2.4) that

$$\rho_r \rho_l (v_r - v_l) \left( (v_r - \frac{A}{\rho_r}) - (v_l - \frac{A}{\rho_l}) \right) = 0, \quad (2.5)$$

from which we have  $v_r = v_l$  or  $v_r - \frac{A}{\rho_r} = v_l - \frac{A}{\rho_l}$ .

Thus, the two states  $(\rho_r, v_r)$  and  $(\rho_l, v_l)$  can be connected by a 1-contact discontinuity if and only if

$$J_1 : \quad \sigma(t) = v_r + \beta t - \frac{A}{\rho_r} = v_l + \beta t - \frac{A}{\rho_l}, \quad (2.6)$$

and can be connected by a 2-contact discontinuity if and only if

$$J_2 : \quad \sigma(t) = v_r + \beta t = v_l + \beta t. \quad (2.7)$$

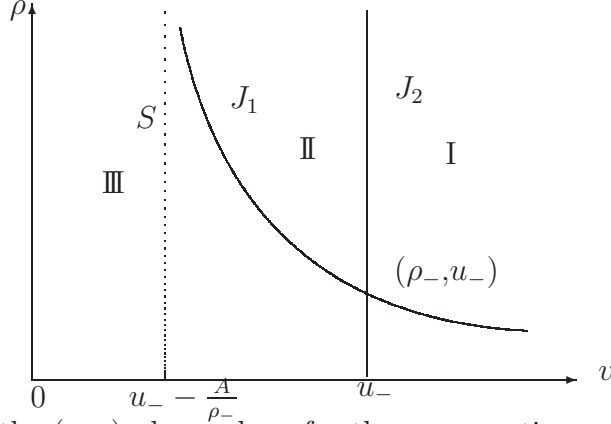


Fig.1 the  $(\rho, v)$  phase plane for the conservative system (2.1).

In the  $(\rho, v)$  phase plane, for the given state  $(\rho_-, u_-)$ , it follow from (2.6) that the sets of states connected on the right consists of the 1-contact discontinuity curve  $J_1(\rho_-, u_-)$  satisfying  $v - \frac{A}{\rho} = u_- - \frac{A}{\rho_-}$ , which has two asymptotes  $S : v = u_- - \frac{A}{\rho_-}$  and  $\rho = 0$ . Similarly, it follow from (2.7) that the sets of states connected on the right consists of the 2-contact discontinuity curve  $J_2(\rho_-, u_-)$  satisfying  $v = u_-$ . In the  $(\rho, v)$  phase plane with  $\rho > 0, v \geq 0$ , let us draw Fig.1 to depict these curves together which divide the  $(\rho, v)$  phase plane into three parts I, II and III, where

$$\begin{aligned} \text{I} &= \{(\rho, v) | v \geq u_-\}, \\ \text{II} &= \{(\rho, v) | u_- - \frac{A}{\rho_-} < v < u_-\}, \\ \text{III} &= \{(\rho, v) | v \leq u_- - \frac{A}{\rho_-}\}. \end{aligned}$$

When  $(\rho_+, u_+) \in \text{I} \cup \text{II}$ , namely  $u_+ > u_- - \frac{A}{\rho_-}$ , the Riemann solutions consists of two contact discontinuity  $J_1$  and  $J_2$  with the intermediate constant state  $(\rho_*, v_*)$  between them besides constant states  $(\rho_-, u_-)$  and  $(\rho_+, u_+)$ , where

$$\begin{cases} v_* - \frac{A}{\rho_*} = u_- - \frac{A}{\rho_-}, \\ u_+ = v_*. \end{cases} \quad (2.8)$$

which immediately leads to

$$\left(\frac{A}{\rho_*}, v_*\right) = \left(u_+ - u_- + \frac{A}{\rho_-}, u_+\right). \quad (2.9)$$

The propagation speed of  $J_1$  and  $J_2$  are given by  $\sigma_1(t) = u_- - \frac{A}{\rho_-} + \beta t$  and  $\sigma_2(t) = u_+ + \beta t$ , respectively.

On the other hand, when  $0 \leq (\rho_+, u_+) \in \text{III}$ , namely  $u_+ \leq u_- - \frac{A}{\rho_-}$ , then the characteristic curves for the Riemann problem (2.1) and (2.2) overlap in a domain  $\Omega$  such that singularity will happen in  $\Omega$ . For completeness, we simply compute the characteristic curves emitting from the origin  $(0, 0)$  which are determined by

$$\frac{dx_i^\pm(t)}{dt} = \lambda_i(\rho_\pm, u_\pm).$$

Thus, we have

$$\begin{aligned} x_1^-(t) &= (u_- - \frac{A}{\rho_-})t + \frac{1}{2}\beta t^2, & x_1^+(t) &= (u_+ - \frac{A}{\rho_+})t + \frac{1}{2}\beta t^2, \\ x_2^-(t) &= u_-t + \frac{1}{2}\beta t^2, & x_2^+(t) &= u_+t + \frac{1}{2}\beta t^2. \end{aligned}$$

Let us draw Fig.2 to explain this phenomenon in detail. In fact, the Cauchy problem for the Chaplygin pressure Aw-Rascle model has been well investigated by us [48] recently by using the generalized characteristic method.

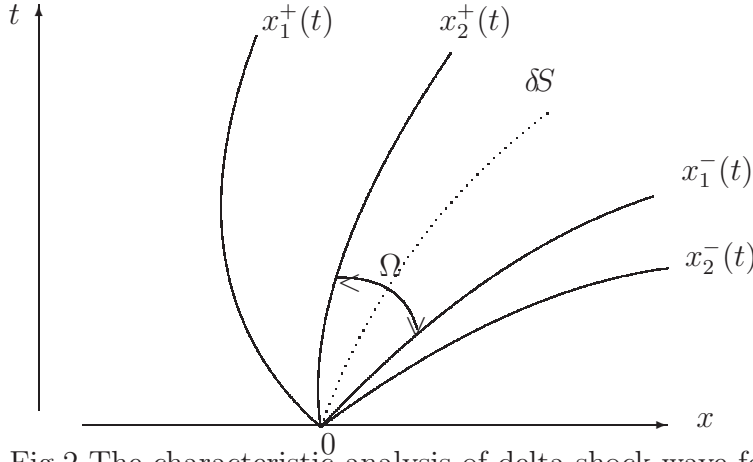


Fig.2 The characteristic analysis of delta shock wave for the Riemann problem (2.1) and (2.2) when  $u_+ < u_- - \frac{A}{\rho_-}$ .

The formation of singularity for the solution to Riemann problem (2.1) and (2.2) is due to the overlap of linearly degenerate characteristics. Thus, the nonclassical situation appears for some certain initial data where the Cauchy problem usually does not own a weak  $L^\infty$ -solution. In order to solve the Riemann problem (2.1) and (2.2) in the framework of nonclassical solutions, motivated by [28], a solution containing a weighted  $\delta$ -measure supported on a curve should be introduced.

**Definition 2.1.** To define the measure solutions, a two-dimensional weighted  $\delta$ -measure  $p(s)\delta_S$  supported on a smooth curve  $S = \{(x(s), t(s)) : a < s < b\}$  can be defined as

$$\langle p(s)\delta_S, \psi(x(s), t(s)) \rangle = \int_a^b p(s)\psi(x(s), t(s))\sqrt{x'(s)^2 + t'(s)^2}ds, \quad (2.10)$$

for any  $\psi \in C_0^\infty(R \times R_+)$ .

For convenience, we usually select the parameter  $s = t$  and use  $w(t) = \sqrt{1 + x'(t)^2}p(t)$  to denote the strength of delta shock wave from now on. In what follows, let us provide the definition of delta shock wave solution to the Riemann problem (2.1) and (2.2) in the framework introduced by Danilov and Shelkovich [12, 13] and developed by Kalisch and Mitrovic [21, 22].

Let us suppose that  $\Gamma = \{\gamma_i \mid i \in I\}$  is a graph in the upper half plane  $\{(x, t) \mid x \in R, t \in [0, +\infty)\}$ , which contains Lipschitz continuous arcs  $\gamma_i$  where  $i \in I$  and  $I$  is a finite index set. Let  $I_0$  be a subset of  $I$  which contains all indices of arcs starting from the  $x$ -axis. Let us use  $\Gamma_0 = \{x_j^0 \mid j \in I_0\}$  to denote the set of initial points of the arcs  $\gamma_j$  with  $j \in I_0$ . Then, one can define the solutions in the sense of distributions to Cauchy problem for the conservative system (2.1) with delta measure initial data below.



**Definition 2.2.** Let  $(\rho, v)$  be a pair of distributions where  $\rho$  is represented in the form

$$\rho(x, t) = \hat{\rho}(x, t) + w(x, t)\delta(\Gamma), \quad (2.11)$$

in which  $\hat{\rho}, v \in L^\infty(R \times R_+)$  and the singular part is defined by

$$w(x, t)\delta(\Gamma) = \sum_{i \in I} w_i(x, t)\delta(\gamma_i). \quad (2.12)$$

Let us consider the delta shock wave type initial data of the form

$$(\rho, v)(x, 0) = (\hat{\rho}(x) + \sum_{j \in I_0} w_j(x_j^0, 0)\delta(x - x_j^0), v_0(x)). \quad (2.13)$$

in which  $\hat{\rho}_0(x), v_0(x) \in L^\infty(R)$ , then the above pair of distributions  $(\rho, v)$  are called as a generalized delta shock wave solution to the conservative system (2.1) with initial data (2.13) if the following integral identities

$$\begin{aligned} & \int_{R_+} \int_R (\hat{\rho}\psi_t + \hat{\rho}(v + \beta t)\psi_x) dx dt + \sum_{i \in I} \int_{\gamma_i} w_i(x, t) \frac{\partial \psi(x, t)}{\partial l} dl \\ & + \int_R \hat{\rho}_0(x)\psi(x, 0) dx + \sum_{k \in I_0} w_k(x_k^0, 0)\psi(x_k^0, 0) = 0, \end{aligned} \quad (2.14)$$

$$\begin{aligned} & \int_{R_+} \int_R (\hat{\rho}(v + P)\psi_t + \hat{\rho}(v + P)(v + \beta t)\psi_x) dx dt + \sum_{i \in I} \int_{\gamma_i} w_i(x, t)v_\delta(x, t) \frac{\partial \psi(x, t)}{\partial l} dl \\ & + \int_R \hat{\rho}_0(x)v_0(x)\psi(x, 0) dx + \sum_{k \in I_0} w_k(x_k^0, 0)v_\delta(x_k^0, 0)(\psi(x_k^0, 0)) = 0, \end{aligned} \quad (2.15)$$

hold for any test function  $\psi \in C_c^\infty(R \times R_+)$ , in which  $\frac{\partial \psi(x, t)}{\partial l}$  stands for the tangential derivative of a function  $\psi$  on the graph  $\gamma_i$  and  $\int_{\gamma_i}$  is the line integral along the arc  $\gamma_i$ .

The above-defined singular solution should be understood in the sense of weak asymptotic solutions. More precisely, let  $f_\epsilon(x) \in D'(R)$  be a family of distributions depending on  $\epsilon \in (0, 1)$ , then we have  $f_\epsilon(x) = o_{D'}(1)$  if the estimate  $\langle f_\epsilon, \psi \rangle = o(1)$  as  $\epsilon \rightarrow 0$  holds for any  $\psi \in D(R)$ . Then, the family of pairs of functions  $(\rho_\epsilon, v_\epsilon)$  is called a weak asymptotic solution of the Cauchy problem (2.1) and (2.12) if the limit of  $\epsilon \rightarrow 0$  of  $(\rho_\epsilon, v_\epsilon)$  is a pair of distributions for every fixed  $t \in R_+$ , where

$$\begin{cases} (\rho_\epsilon)_t + (\rho_\epsilon(v_\epsilon + \beta t))_x = o_{D'}(1), \\ (\rho_\epsilon(v_\epsilon + P_\epsilon))_t + (\rho_\epsilon(v_\epsilon + P_\epsilon)(v_\epsilon + \beta t))_x = o_{D'}(1), \end{cases} \quad (2.16)$$

and

$$\rho_\epsilon|_{t=0} - \rho(x, 0) = o_{D'}(1), \quad v_\epsilon|_{t=0} - v(x, 0) = o_{D'}(1) \quad \text{as } \epsilon \rightarrow 0. \quad (2.17)$$

It can be seen from [12, 13] that the limit  $(\rho_\epsilon, v_\epsilon)$  as  $\epsilon \rightarrow 0$  can be understood in Definition 2.2. The weak asymptotic solution is constructed such that the terms that do not have a distributional limit cancel in the limit  $\epsilon \rightarrow 0$  and the problem about multiplication of distributions is automatically eliminated.



With the above definition, if  $(\rho_+, u_+) \in \mathbb{III}$  and  $u_+ < u_- - \frac{A}{\rho_-}$ , we consider solutions of the form

$$(\rho, v)(x, t) = \begin{cases} (\rho_-, u_-), & x < x(t), \\ (w(t)\delta(x - x(t)), v_\delta), & x = x(t), \\ (\rho_+, u_+), & x > x(t), \end{cases} \quad (2.18)$$

where  $x(t)$ ,  $w(t)$  and  $\sigma(t) = x'(t)$  denote respectively the location, weight and propagation speed of the delta shock,  $v_\delta$  indicates the assignment of  $v$  on this delta shock wave, and  $\frac{1}{\rho}$  is equal to zero on this delta shock wave. In fact, the delta shock wave solution (2.18) to the the Riemann problem (2.1) and (2.2) is the simplest example that the graph  $\Gamma$  contains only one arc. When  $u_+ = u_- - \frac{A}{\rho_-}$ , it can be discussed similarly and we omit it.

Let us check briefly that the delta shock wave solution of the form (2.18) to the the Riemann problem (2.1) and (2.2) satisfy the following generalized Rankine-Hugoniot conditions

$$\begin{cases} \frac{dx(t)}{dt} = \sigma(t) = v_\delta + \beta t, \\ \frac{dw(t)}{dt} = \sigma(t)[\rho] - [\rho(v + \beta t)], \\ \frac{d(w(t)v_\delta)}{dt} = \sigma(t)[\rho(v - \frac{A}{\rho})] - [\rho(v - \frac{A}{\rho})(v + \beta t)]. \end{cases} \quad (2.19)$$

Let us assume that the delta shock wave curve  $\Gamma : (x, t)|x = x(t)$  is a smooth curve in the  $(x, t)$  plane across which  $(\rho, v)$  is a jump discontinuity. Let  $P$  be any point on  $\Gamma$  and let  $\Omega$  be a small ball centered at the point  $P$ . Then, we make a step further to assume that the intersection point of  $\Omega$  and  $\Gamma$  are  $P_1 = (x(t_1), t_1)$  and  $P_2 = (x(t_2), t_2)$  where  $t_1 < t_2$ , and  $\Omega_-$  and  $\Omega_+$  are the left-hand and right-hand parts of  $\Omega$  cut by  $\Gamma$  respectively. Then, for any test function  $\psi \in C_c^\infty(\Omega)$ , by applying the divergence theorem, we have

$$\begin{aligned} I &= \int \int_{\Omega} \left( \rho(v - \frac{A}{\rho})\psi_t + \rho(v - \frac{A}{\rho})(v + \beta t)\psi_x \right) dx dt \\ &= \int \int_{\Omega_-} \left( \rho_-(u_- - \frac{A}{\rho_-})\psi_t + \rho_-(u_- - \frac{A}{\rho_-})(u_- + \beta t)\psi_x \right) dx dt \\ &\quad + \int \int_{\Omega_+} \left( \rho_+(u_+ - \frac{A}{\rho_+})\psi_t + \rho_+(u_+ - \frac{A}{\rho_+})(u_+ + \beta t)\psi_x \right) dx dt \\ &\quad + \int_{t_1}^{t_2} w(t) \left( v_\delta \psi_t(x(t), t) + v_\delta(v_\delta + \beta t)\psi_x(x(t), t) \right) dt \\ &= \int_{\partial\Omega_-} -\rho_-(u_- - \frac{A}{\rho_-})\psi dx + \rho_-(u_- - \frac{A}{\rho_-})(u_- + \beta t)\psi dt \\ &\quad + \int_{\partial\Omega_+} -\rho_+(u_+ - \frac{A}{\rho_+})\psi dx + \rho_+(u_+ - \frac{A}{\rho_+})(u_+ + \beta t)\psi dt \\ &\quad + \int_{t_1}^{t_2} w(t) \left( v_\delta \psi_t(x(t), t) + v_\delta(v_\delta + \beta t)\psi_x(x(t), t) \right) dt \end{aligned}$$

$$\begin{aligned}
&= \int_{t_1}^{t_2} \left( \left( \rho_+ \left( u_+ - \frac{A}{\rho_+} \right) - \rho_- \left( u_- - \frac{A}{\rho_-} \right) \right) \frac{dx}{dt} \right. \\
&\quad \left. + \rho_- \left( u_- - \frac{A}{\rho_-} \right) (u_- + \beta t) - \rho_+ \left( u_+ - \frac{A}{\rho_+} \right) (u_+ + \beta t) \right) \psi(x(t), t) dt \\
&\quad \int_{t_1}^{t_2} w(t) v_\delta d\psi(x(t), t).
\end{aligned}$$

Thus, one can see that the third equality in (2.19) is satisfied when  $I$  vanishes for any  $\psi \in C_c^\infty(\Omega)$ . In the same way as above, we can check that the second identity holds. Thus, the proof is complete.

In order to ensure uniqueness, it should also satisfy an over-compressive entropy condition for the delta shock wave as follows:

$$\lambda_1(\rho_+, u_+) < \lambda_2(\rho_+, u_+) < \sigma(t) < \lambda_1(\rho_-, u_-) < \lambda_2(\rho_-, u_-) \quad (2.20)$$

which enables us to have

$$0 \leq u_+ < v_\delta < u_- - \frac{A}{\rho_-}. \quad (2.21)$$

The generalized Rankine-Hugoniot conditions (2.19) reflect the relationship among the location, weight and propagation speed of delta shock wave. The entropy condition (2.20) for the delta shock wave is an over-compressive condition which implies that all the characteristics on both sides of the delta shock are incoming.

It follows from (2.19) that

$$\frac{dw(t)}{dt} = v_\delta(\rho_+ - \rho_-) - (\rho_+ u_+ - \rho_- u_-), \quad (2.22)$$

$$v_\delta \frac{dw(t)}{dt} = v_\delta(\rho_+ u_+ - \rho_- u_-) - (\rho_+ u_+^2 - \rho_- u_-^2) + A(u_+ - u_-), \quad (2.23)$$

Thus, we have

$$(\rho_+ - \rho_-)v_\delta^2 - 2(\rho_+ u_+ - \rho_- u_-)v_\delta + (\rho_+ u_+^2 - \rho_- u_-^2) - A(u_+ - u_-) = 0, \quad (2.24)$$

For convenience, let us denote

$$w_0 = \sqrt{\rho_+ \rho_- (u_+ - u_-) \left( (u_+ - u_-) - \left( \frac{A}{\rho_+} - \frac{A}{\rho_-} \right) \right)}, \quad (2.25)$$

If  $\rho_+ \neq \rho_-$ , with the entropy condition (2.20) in mind, one can obtain directly from (2.24) that

$$v_\delta = \frac{\rho_+ u_+ - \rho_- u_- + w_0}{\rho_+ - \rho_-}, \quad (2.26)$$

which enables us to get

$$\sigma(t) = v_\delta + \beta t, \quad x(t) = v_\delta t + \frac{1}{2} \beta t^2 \quad w(t) = w_0 t, \quad (2.27)$$

Otherwise, if  $\rho_+ = \rho_-$ , then we have

$$v_\delta = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-}). \quad (2.28)$$

In this particular case, we can also get

$$\sigma(t) = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-}) + \beta t, \quad x(t) = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-})t + \frac{1}{2}\beta t^2, \quad w(t) = (\rho_- u_- - \rho_+ u_+)t. \quad (2.29)$$

### 3. Riemann problem for the original system

In this section, let us return to the Riemann problem (1.1) and (1.2). If  $(\rho_+, u_+) \in \text{I} \cup \text{II}$ , namely  $u_+ > u_- - \frac{A}{\rho_-}$ , the Riemann solutions to (1.1) and (1.2) can be represented as

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_- + \beta t), & x < x_1(t), \\ (\rho_*, v_* + \beta t), & x_1(t) < x < x_2(t), \\ (\rho_+, u_+ + \beta t), & x > x_2(t), \end{cases} \quad (3.1)$$

where  $(\rho_*, v_*)$  is given by (2.9) and the position of the two contact discontinuities  $J_1$  and  $J_2$  are given respectively by

$$x_1(t) = (u_- - \frac{A}{\rho_-})t + \frac{1}{2}\beta t^2, \quad x_2(t) = u_+ t + \frac{1}{2}\beta t^2. \quad (3.2)$$

Let us draw Fig.3 to illustrate this situation in detail.

Analogously, if  $(\rho_+, u_+) \in \text{III}$ , namely  $0 \leq u_+ \leq u_- - \frac{A}{\rho_-}$ , then we can also define the weak solutions in the sense of distributions to the Riemann problem (1.1) and (1.2) below.

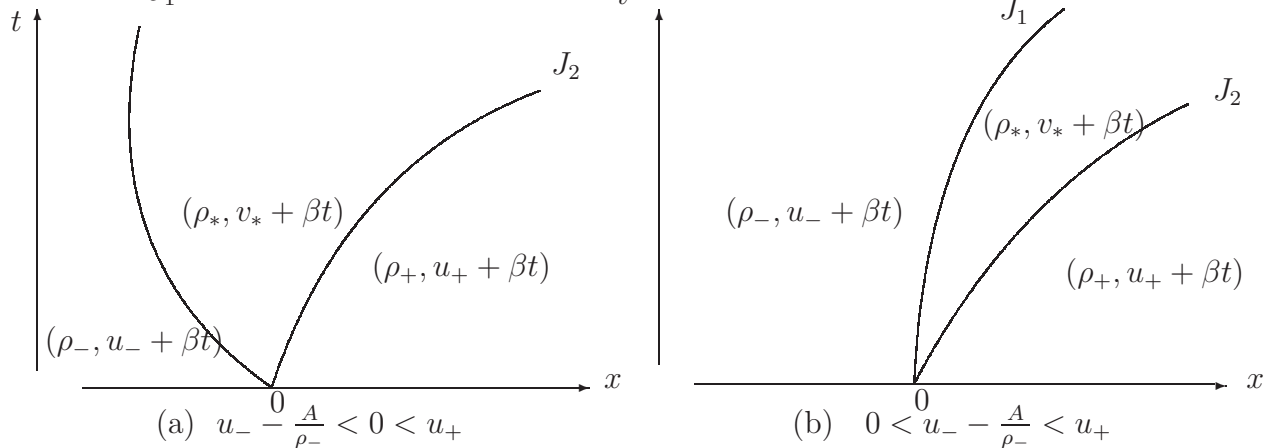


Fig.3 The Riemann solution to (1.1) and (1.2) when  $u_- - \frac{A}{\rho_-} < u_+$  and  $\beta > 0$ , where  $(\rho_*, v_*)$  is given by (2.9).

**Definition 3.1.** Let  $(\rho, u)$  be a pair of distributions in which  $\rho$  has the form of (2.11), then it is called as the delta shock wave solution to the Riemann problem (1.1) and (1.2) if it satisfies

$$\begin{cases} \langle \rho, \psi_t \rangle + \langle \rho u, \psi_x \rangle = 0, \\ \langle \rho(u + P), \psi_t \rangle + \langle \rho u(u + P), \psi_x \rangle = -\langle \beta \rho, \psi \rangle, \end{cases} \quad (3.3)$$

for any  $\psi \in C_0^\infty(R \times R^+)$ , in which

$$\langle \rho u(u + P), \psi \rangle = \int_0^\infty \int_{-\infty}^\infty (\widehat{\rho} u(u - \frac{1}{\widehat{\rho}})) \psi dx dt + \langle w(t)(u_\delta(t))^2 \delta_S \psi \rangle,$$

and  $u_\delta(t)$  is the assignment of  $u$  on this delta shock wave curve.

With the above definition in mind, if  $u_+ < u_- - \frac{A}{\rho_-}$  is satisfied, then we look for a piecewise smooth solution to the Riemann problem (1.1) and (1.2) in the form

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_- + \beta t), & x < x(t), \\ (w(t)\delta(x - x(t)), u_\delta(t)), & x = x(t), \\ (\rho_+, u_+ + \beta t), & x > x(t), \end{cases} \quad (3.4)$$

It is worthwhile to notice that  $u_\delta(t) - \beta t$  is assumed to be a constant based on the result in Sect.2. With the similar analysis and derivation as before, the delta shock wave solution of the form (3.4) to the Riemann problem (1.1) and (1.2) should also satisfy the following generalized Rankine-Hugoniot conditions

$$\begin{cases} \frac{dx(t)}{dt} = \sigma(t) = u_\delta(t), \\ \frac{dw(t)}{dt} = \sigma(t)[\rho] - [\rho u], \\ \frac{d(w(t)u_\delta(t))}{dt} = \sigma(t)[\rho(u - \frac{A}{\rho})] - [\rho u(u - \frac{A}{\rho})] + \beta w(t). \end{cases} \quad (3.5)$$

in which the jumps across the discontinuity are

$$[\rho u] = \rho_+(u_+ + \beta t) - \rho_-(u_- + \beta t), \quad (3.6)$$

$$[\rho u(u - \frac{A}{\rho})] = \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+}) - \rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-}). \quad (3.7)$$

In order to ensure the uniqueness to the Riemann problem (1.1) and (1.2), the over-compressive entropy condition for the delta shock wave

$$u_+ + \beta t < u_\delta(t) < u_- - \frac{A}{\rho_-} + \beta t. \quad (3.8)$$

should also be proposed when  $0 \leq u_+ < u_- - \frac{A}{\rho_-}$ .

Like as before, we can also obtain  $x(t)$ ,  $\sigma(t)$  and  $w(t)$  from (3.5) and (3.8) together. In brief, we have the following theorem to depict the Riemann solution to (1.1) and (1.2) when the Riemann initial data (1.2) satisfy  $0 \leq u_+ < u_- - \frac{A}{\rho_-}$  and  $\rho_+ \neq \rho_-$ .

**Theorem 3.2.** *If both  $0 \leq u_+ < u_- - \frac{A}{\rho_-}$  and  $\rho_+ \neq \rho_-$  are satisfied, then the delta shock solution to the Riemann solutions to (1.1) and (1.2) can be expressed as*

$$\begin{cases} \frac{dx(t)}{dt} = \sigma(t) = u_\delta(t), \\ \frac{dw(t)}{dt} = \sigma(t)[\rho] - [\rho u], \\ \frac{d(w(t)u_\delta(t))}{dt} = \sigma(t)[\rho(u - \frac{A}{\rho})] - [\rho u(u - \frac{A}{\rho})] + \beta w(t). \end{cases} \quad (3.9)$$

in which

$$\sigma(t) = u_\delta(t) = v_\delta + \beta t, \quad x(t) = v_\delta t + \frac{1}{2}\beta t^2 \quad w(t) = w_0 t, \quad (3.10)$$

in which  $w_0$  and  $v_\delta$  are given by (2.21) and (2.22) respectively.

Let us check briefly that the above constructed delta shock wave solution (3.9) and (3.10) should satisfy (1.1) in the sense of distributions. The proof of this theorem is completely analogous to those in [32, 33]. Therefore, we only deliver the main steps for the proof of the second equality in (3.3) for completeness. Actually, one can deduce that

$$\begin{aligned} I &= \int_0^\infty \int_{-\infty}^\infty (\rho(u - \frac{A}{\rho})\psi_t + \rho u(u - \frac{A}{\rho})\psi_x) dx dt \\ &= \int_0^\infty \int_{-\infty}^{x(t)} (\rho_-(u_- + \beta t - \frac{A}{\rho_-})\psi_t + \rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-})\psi_x) dx dt \\ &\quad + \int_0^\infty \int_{x(t)}^\infty (\rho_+(u_+ + \beta t - \frac{A}{\rho_+})\psi_t + \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+})\psi_x) dx dt \\ &\quad + \int_0^\infty w_0 t(v_\delta + \beta t)(\psi_t(x(t), t) + (v_\delta + \beta t)\psi_x(x(t), t)) dt. \end{aligned}$$

It can be derived from (3.10) that the curve of delta shock wave is given by

$$x(t) = v_\delta t + \frac{1}{2}\beta t^2. \quad (3.11)$$

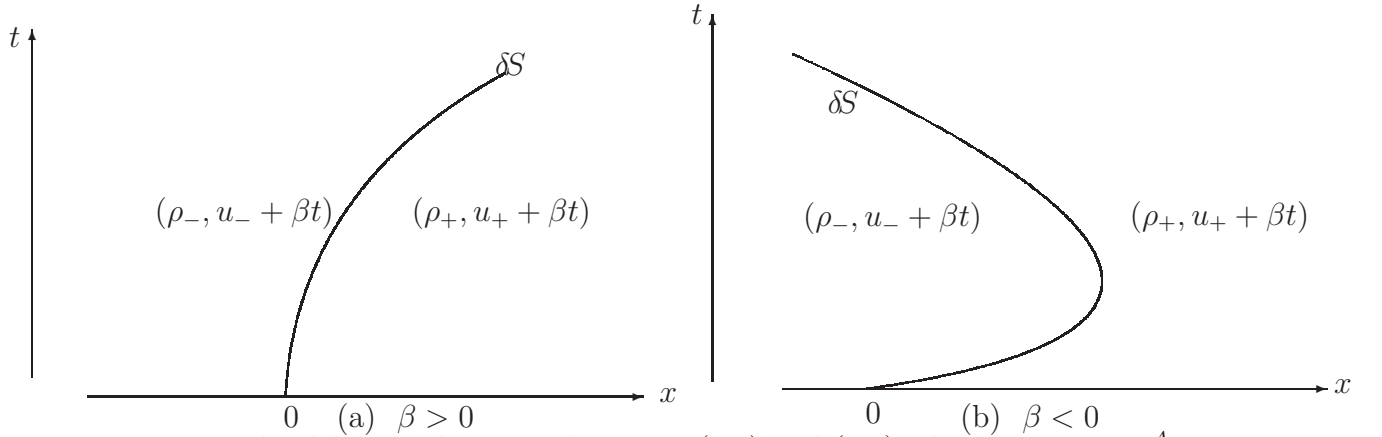


Fig.4 The delta shock wave solution to (1.1) and (1.2) when  $u_+ < u_- - \frac{A}{\rho_-}$ , where  $v_\delta > 0$  is given by (2.26) for  $\rho_- \neq \rho_+$  and (2.28) for  $\rho_- = \rho_+$ .

For  $\beta > 0$  (Fig.4a), there exists an inverse function of  $x(t)$  globally in the time  $t$ , which may be written in the form

$$t(x) = \sqrt{\frac{v_\delta^2}{\beta^2} + \frac{2x}{\beta}} - \frac{v_\delta}{\beta}.$$

Otherwise, for  $\beta < 0$  (Fig.4b), there is a critical point  $(-\frac{v_\delta^2}{2\beta}, -\frac{v_\delta}{\beta})$  on the delta shock wave curve such that  $x'(t)$  change its sign when across the critical point. Thus, the inverse function of  $x(t)$  is needed to find respectively for  $t \leq -\frac{v_\delta}{\beta}$  and  $t > -\frac{v_\delta}{\beta}$ , which enable us

to have

$$t(x) = \begin{cases} -\sqrt{\frac{v_\delta^2}{\beta^2} + \frac{2x}{\beta}} - \frac{v_\delta}{\beta}, & t \leq -\frac{v_\delta}{\beta}, \\ \sqrt{\frac{v_\delta^2}{\beta^2} + \frac{2x}{\beta}} - \frac{v_\delta}{\beta}, & t > -\frac{v_\delta}{\beta}. \end{cases}$$

Without loss of generality, let us assume that  $\beta > 0$  for simplicity. Actually, the other situation can be dealt with similarly. Under our assumption, it follows from (3.12) that the position of delta shock wave satisfies  $x = x(t) > 0$  for all the time. It follows from (3.10) that

$$\begin{aligned} \frac{d\psi(x(t), t)}{dt} &= \psi_t(x(t), t) + \frac{dx(t)}{dt} \psi_x(x(t), t) \\ &= \psi_t(x(t), t) + (v_\delta + \beta t) \psi_x(x(t), t) \\ &= \psi_t(x(t), t) + u_\delta(t) \psi_x(x(t), t). \end{aligned}$$

By exchanging the ordering of integrals and using integration by parts, we have

$$\begin{aligned} I &= \int_0^\infty \int_{t(x)}^\infty \rho_-(u_- + \beta t - \frac{A}{\rho_-}) \psi_t dt dx + \int_0^\infty \int_{t(x)}^\infty \rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-}) \psi_x dt dx \\ &\quad + \int_0^\infty \int_0^{t(x)} \rho_+(u_+ + \beta t - \frac{A}{\rho_+}) \psi_t dt dx + \int_0^\infty \int_0^{t(x)} \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+}) \psi_x dt dx \\ &\quad + \int_0^\infty w_0 t (v_\delta + \beta t) d\psi(x(t), t) \\ &= \int_0^\infty (\rho_+(u_+ + \beta t(x) - \frac{A}{\rho_+}) - \rho_-(u_- + \beta t(x) - \frac{A}{\rho_-})) \psi(x, t(x)) dx \\ &\quad + \int_0^\infty (\rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-}) - \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+})) \psi(x(t), t) dt \\ &\quad - \int_0^\infty \int_{t(x)}^\infty \beta \rho_- \psi dt dx - \int_0^\infty \int_0^{t(x)} \beta \rho_+ \psi dt dx - \int_0^\infty w_0 (v_\delta + 2\beta t) \psi(x(t), t) dt \\ &= \int_0^\infty A(t) \psi(x(t), t) dt - \beta (\int_0^\infty \int_{-\infty}^{x(t)} \rho_- \psi dx dt + \int_0^\infty \int_{x(t)}^\infty \rho_+ \psi dx dt), \end{aligned} \tag{3.12}$$

in which

$$\begin{aligned} C(t) &= (\rho_+(u_+ + \beta t - \frac{A}{\rho_+}) - \rho_-(u_- + \beta t - \frac{A}{\rho_-}))(v_\delta + \beta t) \\ &\quad + (\rho_-(u_- + \beta t)(u_- + \beta t - \frac{A}{\rho_-}) - \rho_+(u_+ + \beta t)(u_+ + \beta t - \frac{A}{\rho_+})) \\ &\quad - w_0 (v_\delta + 2\beta t). \end{aligned}$$

By a tedious calculation, we have

$$A(t) = -\beta w_0 t = -\beta w(t). \tag{3.13}$$

Thus, it can be concluded from (3.12) and (3.13) together that the second equality in (3.3) holds in the sense of distributions. The proof is completed.

*Remark 3.1.* If both  $0 \leq u_+ < u_- - \frac{A}{\rho_-}$  and  $\rho_+ = \rho_-$  are satisfied, then the delta shock solution to the Riemann solutions to (1.1) and (1.2) can be expressed in the form (3.11) where

$$\sigma(t) = u_\delta(t) = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-}) + \beta t, \quad x(t) = \frac{1}{2}(u_+ + u_- - \frac{A}{\rho_-})t + \frac{1}{2}\beta t^2, \quad w(t) = (\rho_- u_- - \rho_+ u_+)t. \quad (3.14)$$

The process of proof is completely similar and we omit the details.

*Remark 3.2.* If  $u_+ = u_- - \frac{A}{\rho_-}$ , then the delta shock solution to the Riemann solutions to (1.1) and (1.2) can be also expressed as the form in Theorem 3.2 and Remark 3.1. The process of proof is easy and we omit the details.

#### 4. The vanishing pressure limit of Riemann solutions to (1.1) and (1.2)

In this section, we consider the vanishing pressure limit of Riemann solutions to (1.1) and (1.2). According to the relations between  $u_-$  and  $u_+$ , we will divide our discussion into the following three case:

$$(1) \ u_- < u_+; \quad (2) \ u_- = u_+; \quad (3) \ u_- > u_+.$$

**Case 4.1.**  $u_- < u_+$

In this case,  $(\rho_+, u_+) \in \text{I}$  in the  $(\rho, v)$  plane, so the Riemann solutions to (1.1) and (1.2) is given by (3.1) and (3.2), where  $(\rho_*, v_*)$  is given by (2.9). From (2.9) we have

$$\lim_{A \rightarrow 0} \rho_* = \lim_{A \rightarrow 0} \frac{A}{u_+ - u_- + \frac{A}{\rho_-}} = 0,$$

which indicates the occurrence of the vacuum states. Furthermore, the Riemann solutions to (1.1) and (1.2) converge to

$$\lim_{A \rightarrow 0} (\rho, u)(x, t) = \begin{cases} (\rho_-, u_- + \beta t), & x < u_- t + \frac{1}{2}\beta t^2, \\ \text{vacuum}, & u_- t + \frac{1}{2}\beta t^2 < x < u_+ t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > u_+ t + \frac{1}{2}\beta t^2, \end{cases} \quad (4.1)$$

which is exactly the corresponding Riemann solutions to the transportation equations with the same source term and the same initial data [32].

**Case 4.2.**  $u_- = u_+$

In this case,  $(\rho_+, u_+)$  is on the  $J_2$  curve in the  $(\rho, v)$  plane, so the Riemann solutions to (1.1) and (1.2) is given as follows:

$$(\rho, u)(x, t) = \begin{cases} (\rho_-, u_- + \beta t), & x < u_- t + \frac{1}{2}\beta t^2, \\ (\rho_+, u_+ + \beta t), & x > u_+ t + \frac{1}{2}\beta t^2, \end{cases} \quad (4.2)$$

which is exactly the corresponding Riemann solutions to the transportation equations with the same source term and the same initial data [32].

**Case 4.3.**  $u_- > u_+$

**Lemma 4.1.** *If  $u_- > u_+$ , there exist  $A_1 > A_0 > 0$ , such that  $(\rho_+, u_+) \in \text{II}$  as  $A_0 < A < A_1$ , and  $(\rho_+, u_+) \in \text{III}$  as  $A \leq A_0$ .*



**Proof.** If  $(\rho_+, u_+) \in \mathbb{II}$ , then  $0 < u_- - \frac{A}{\rho_-} < u_+ < u_-$ , which gives  $\rho_- u_- > A > \rho_-(u_- - u_+)$ . Thus we take  $A_0 = \rho_-(u_- - u_+)$  and  $A_0 = \rho_- u_-$ , then  $(\rho_+, u_+) \in \mathbb{II}$  as  $A_0 < A < A_1$  and  $(\rho_+, u_+) \in \mathbb{III}$  as  $A \leq A_0$ .

When  $A_0 < A < A_1$ ,  $(\rho_+, u_+) \in \mathbb{II}$  in the  $(\rho, v)$  plane, so the Riemann solutions to (1.1) and (1.2) is given by (3.1) and (3.2), where  $(\rho_*, v_*)$  is given by (2.9). From (2.9) we have From (2.9) we have

$$\lim_{A \rightarrow A_0} \rho_* = \lim_{A \rightarrow A_0} \frac{A}{u_+ - u_- + \frac{A}{\rho_-}} = \lim_{A \rightarrow A_0} \frac{\rho_- A}{A - A_0} = \infty.$$

Furthermore, we have the following result.

**Lemma 4.2.** Let  $\frac{dx_1(t)}{dt} = \sigma_1(t)$ ,  $\frac{dx_2(t)}{dt} = \sigma_2(t)$ , then we have

$$\lim_{A \rightarrow A_0} v_* + \beta t = \lim_{A \rightarrow A_0} \sigma_1(t) = \lim_{A \rightarrow A_0} \sigma_2(t) = (u_- - \frac{A_0}{\rho_-})t + \beta t = u_+ + \beta t =: \sigma(t), \quad (4.3)$$

$$\lim_{A \rightarrow A_0} \int_{x_1(t)}^{x_2(t)} \rho_* dx = A_0 t, \quad (4.4)$$

$$\lim_{A \rightarrow A_0} \int_{x_1(t)}^{x_2(t)} \rho_*(v_* + \beta t) dx = (u_+ + \beta t) A_0 t. \quad (4.5)$$

**Proof.** (4.3) is obviously true. We will only prove (4.4) and (4.5).

$$\lim_{A \rightarrow A_0} \int_{x_1(t)}^{x_2(t)} \rho_* dx = \lim_{A \rightarrow A_0} \rho_*(x_2(t) - x_1(t)) = \lim_{A \rightarrow A_0} \frac{A}{u_+ - u_- + \frac{A}{\rho_-}} (u_+ - u_- + \frac{A}{\rho_-}) t = A_0 t,$$

$$\lim_{A \rightarrow A_0} \int_{x_1(t)}^{x_2(t)} \rho_*(v_* + \beta t) dx = (u_+ + \beta t) \lim_{A \rightarrow A_0} \int_{x_1(t)}^{x_2(t)} \rho_* dx = (u_+ + \beta t) A_0 t.$$

The proof is completed.

It can be concluded from Lemma 4.2 that the curves of the two contact discontinuities  $J_1$  and  $J_2$  will coincide when  $A$  tends to  $A_0$  and the delta shock waves will form. Next we will arrange the values which gives the exact position, propagation speed and strength of the delta shock wave according to Lemma 4.2.

From (4.4) and (4.5), we let

$$w(t) = A_0 t, \quad (4.6)$$

$$w(t)u_\delta(t) = (u_+ + \beta t) A_0 t, \quad (4.7)$$

then

$$u_\delta(t) = (u_+ + \beta t), \quad (4.8)$$

which is equal to  $\sigma(t)$ . Furthermore, by letting  $\frac{dx(t)}{dt} = \sigma(t)$ , we have

$$x(t) = u_+ t + \frac{1}{2} \beta t^2. \quad (4.9)$$

From (4.6)-(4.9), we can see that the quantities defined above are exactly consistent with those given by (2.25)-(2.29) or (3.10) in which we take  $A = A_0$ . Thus, it uniquely determines that the limits of the Riemann solutions to the system (1.1) and (1.2) when  $A \rightarrow A_0$  in the case  $(\rho_+, u_+) \in \mathbb{II}$  is just the delta shock solution of (1.1) and (1.2) in the case  $(\rho_+, u_+) \in \mathbb{S}$ , where  $\mathbb{S}$  is actually the boundary between the regions  $\mathbb{II}$  and  $\mathbb{III}$ . So we get the following results in the case  $u_+ < u_-$ .

**Theorem 4.1.** *If  $u_+ < u_-$ , for each fixed  $A$  with  $A_0 < A < A_1$ ,  $(\rho_+, u_+) \in \mathbb{II}$  assuming that  $(\rho, u)$  is a solution containing two contact discontinuities  $J_1$  and  $J_2$  of (1.1) and (1.2) which is constructed in Section 3, it is obtained that when  $A \rightarrow A_0$ ,  $(\rho, u)$  converges to a delta shock wave solution of (1.1) and (1.2) when  $A = A_0$ .*

When  $A \leq A_0$ ,  $(\rho_+, u_+) \in \mathbb{III}$ , so the Riemann solutions to (1.1) and (1.2) is given by (3.4) with (3.10) or (3.14), which is a delta shock wave solution. It is easy to see that when  $A \rightarrow 0$ , for  $\rho_+ \neq \rho_-$ ,

$$x(t) \rightarrow \sigma t + \frac{1}{2}\beta t^2, \quad w(t) \rightarrow \sqrt{\rho_+ \rho_-}(u_- - u_+)t, \quad \sigma(t) = u_\delta(t) \rightarrow \sigma + \beta t,$$

where  $\sigma = \frac{\sqrt{\rho_-}u_- + \sqrt{\rho_+}u_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}$ , for  $\rho_+ = \rho_-$ ,

$$x(t) \rightarrow \frac{1}{2}(u_+ + u_-)t + \frac{1}{2}\beta t^2, \quad w(t) \rightarrow \rho_+(u_- - u_+)t, \quad \sigma(t) = u_\delta(t) \rightarrow \frac{1}{2}(u_+ + u_-) + \beta t,$$

which is exactly the corresponding Riemann solutions to the transportation equations with the same source term and the same initial data [32]. Thus, we have the following result:

**Theorem 4.2.** *If  $u_+ < u_-$ , for each fixed  $A < A_0$ ,  $(\rho_+, u_+) \in \mathbb{III}$  assuming that  $(\rho, u)$  is a delta shock wave solution of (1.1) and (1.2) which is constructed in Section 3, it is obtained that when  $A \rightarrow 0$ ,  $(\rho, u)$  converges to a delta shock wave solution to the transportation equations with the same source term and the same initial data [32].*

Now we summarize the main result in this section as follows.

**Theorem 4.3.** *As the pressure vanishes, the Riemann solutions to the Chaplygin pressure Aw-Rascle model with Coulomb-like friction tend to the three kinds of Riemann solutions to the transportation equations with the same source term and the same initial data, which included a delta shock wave and a vacuum state.*

## 5. Conclusions and Discussions

In this work, we have considered the solutions of the Riemann problem for the Chaplygin pressure Aw-Rascle model with Coulomb-like friction in the fully explicit form. In particular, the delta shock wave solution has been discovered in some certain situations, which may be used to explain the serious traffic jam. We find that the Coulomb-like friction term takes the effect to curve the characteristics to the parabolic curves such that the delta shock wave discontinuity is also curved. Thus, the Riemann solutions (1.1) and (1.2) are not self-similar any more. It is worthwhile to note that the Riemann solutions of

(1.1) and (1.2) converge to the corresponding ones of the Chaplygin pressure Aw-Rascle model as  $\beta \rightarrow 0$ , namely the Coulomb-like friction term vanishes. Finally, we analyze the formation of  $\delta$ -shocks and vacuum states in the Riemann solutions in the vanishing pressure limit and show that the Riemann solutions of (1.1) and (1.2) converge to the corresponding ones of the transportation equations with the same source term as the pressure vanishes. These results generalize those obtained in [7, 38] for homogeneous equations to nonhomogeneous equations and are also applicable to the nonsymmetric system of Keyfitz-Kranzer type with the same Chaplygin pressure and Coulomb-like friction.

It is interesting to notice that the above constructed Riemann solutions of (1.1) and (1.2) can be obtained directly from the ones of the Riemann problem for the homogeneous situation by using the change of variables  $x \rightarrow x - \frac{1}{2}\beta t^2$  and  $u \rightarrow u - \beta t$  together, see [23]. It also pointed out in [23] that these solutions are drastically different from each other in that the characteristics are the parabolic curves for the inhomogeneous situation instead of the straight lines for the homogeneous situation. Furthermore, the regions of constant flow are transformed into the regions of constantly accelerated flow and the contact discontinuities and the the delta shock waves bend into parabolic shapes under the influence of the Coulomb-like friction term.

It is worthwhile to note that the method developed in this paper can be used to the inhomogeneous Aw-Rascle model with generalized Chaplygin pressure. Especially, the Aw-Rascle model with generalized Chaplygin pressure has a significant mathematical difference with the Aw-Rascle model with Chaplygin pressure. Thus, it is interesting to study the Riemann problem for the Aw-Rascle model with generalized Chaplygin pressure under the influence of the the Coulomb-like friction term, whose results will also be applicable to the nonsymmetric system of Keyfitz-Kranzer type (1.4) with the same pressure and Coulomb-like friction. We leave this problem for our future work.

## References

- [1] A.Aw and M.Rascle, Resurrection of second order models of traffic flow, *SIAM J. Appl. Math.*, 2000, **60**: 916-938.
- [2] H.Benaoum. Accelerated universe from modified Chaplygin gas and tachyonic fluid. arXiv: hep-th/0205140.
- [3] N.Bilic, G.Tupper and R.Viollier. Unification of dark matter and dark energy: the inhomogeneous Chaplygin gas. *Phys. Lett. B*, 2002, **535**: 17-21.
- [4] F.Bouchut. On zero-pressure gas dynamics//Advances in Kinetic Theory and Computing. Ser Adv Math Appl Sci **22**. River Edge, NJ: World Scientific, 1994: 171-190.
- [5] Y.Brenier. Solutions with concentration to the Riemann problem for one-dimensional Chaplygin gas equations. *J Math Fluid Mech*, 2005, **7**: S326-S331. ““
- [6] S.Chaplygin. On gas jets. *Sci Mem Moscow Univ Math Phys*, 1904, **21**: 1-121.
- [7] G.Q.Chen and H.Liu. Formation of  $\delta$ -shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids. *SIAM J. Math. Anal.* **34** (2003), 925-938.

- [8] H.Cheng. Delta shock waves for a linearly degenerate hyperbolic system of conservation laws of Keyfitz-Kranzer type. *Advances in mathematical Physics*, 2013, Article ID 958120, 10 pages.
- [9] H.Cheng. On a nonsymmetric Keyfitz-Kranzer system of conservation laws with generalized and modified Chaplygin gas pressure law. *Advances in mathematical Physics*, 2013, 14 pages.
- [10] H.Cheng and H.Yang. Approaching Chaplygin pressure limit to the Aw-Rascle model. *J. Math. Anal. Appl.*, 2014, **416**: 839-854.
- [11] C.Daganzo. Requiem for second order fluid approximations of traffic flow. *Transp. Res. Part B*, 1995, **29**: 277-286.
- [12] V.G.Danilov and V.M.Shelkovich. Dynamics of propagation and interaction of  $\delta$ -shock waves in conservation law system. *J. Differential Equations* **221** (2005), 333-381.
- [13] V.G.Danilov and V.M.Shelkovich. Delta-shock waves type solution of hyperbolic systems of conservation laws. *Q. Appl. Math.* **63** (2005), 401-427.
- [14] E W,Rykov Yu G, and Sinai Ya G. Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. *Comm Math Phys*, 1996, **177**: 349-380.
- [15] G.Faccanoni, A.Mangeney. Exact solution for granular flows. *Int. J. Numer. Anal. Mech. Geomech.*, 2012, **37**: 1408-1433.
- [16] V.Gorini, A.Kamenshchik, U.Moschella, V.Pasquier. The Chaplygin gas as a model for dark energy. arXiv: gr-qc/0403062.
- [17] J.Greenberg. Extensions and amplifications of a traffic model of Aw-Rascle. *SIAM J. Appl. Math.*, 2001, **62**(6): 729-745.
- [18] L.Guo. The Riemann problem of the transport equations for the generalized Chaplygin gas. *J. Xinjiang University (Natural Science Edition)*, 2013, **30**(2): 170-176.
- [19] L.Guo, W.Sheng and T.Zhang. The two-dimensional Riemann problem for isentropic Chaplygin gas dynamic system. *Comm Pure Appl Anal*, 2010, **9**(2): 431-458.
- [20] F.Huang and Z.Wang. Well posedness for pressureless flow. *Comm Math Phys*, 2001, **222**: 117-146.
- [21] H.Kalisch and D.Mitrovic. Singular solutions of a fully nonlinear  $2 \times 2$  system of conservation laws. *Proceedings of the Edinburgh Mathematical Society*, 2012, **55**: 711-729.
- [22] H.Kalisch and D.Mitrovic. Singular solutions for shallow water equations. *IMA J. Appl. Math.* 2012, **77**: 340-350.

- [23] K.V.Karelsky, A.S.Petrosyan and S.V.Tarasevich. Nonlinear dynamics of magnethydrodynamic flows of heavy fluid on slope in shallow water approximation. Journal of Experimental and Theoretical Physics. 2014, **146**: 352-367.
- [24] D.Korchinski. Solutions of a Riemann problem for a system of conservation laws possessing no classical weak solution. Thsis, Adelphi University, 1977.
- [25] J.Lebacque, S.Mannar and H.Salem. The Aw-Rascle and Zhangs model: vacuum problems, existence and regularity of the solutions of the Riemann problem. Transp. Res. Part B, 2001, **41**: 710-721.
- [26] J.Li, T.Zhang and S.Yang. The Two-dimensional Riemann Problem in Gas Dynamics. Pitman Monogr Surv Pure Appl Math **98**. Longman Scientific and Technical, 1998.
- [27] Y.-G.Lu. Existence of global entropy solution to general system of Keyfitz-Kranzer type. Journal of Functional Analysis, 2013, **264**: 2457-2468.
- [28] L.Pan and X.Han. The Aw-Rascle traffic model with Chaplygin pressure. J. Math. Anal. Appl., 2013, **401**: 379-387.
- [29] S.Savage and K.Hutter. The motion of finite mass of granular material down a rough incline. J. Fluid Mech., 1989, **199**: 177-215.
- [30] D.Serre. Multidimensional shock interaction for a Chaplygin gas. Arch Rational Mech Anal, 2009, **191**: 539-577.
- [31] M.Setare. Interacting holographic generalized Chaplygin gas model. Phys. Lett. B **654**(2007), 1-6.
- [32] C.Shen. The Riemann problem for the pressureless Euler system with the Coulomb-like friction term. IAM J. Appl. Math., 2015:1-24,doi:10.1039/imamat/hxv028.
- [33] C.Shen. The Riemann problem for the Chaplygin gas equations with a source term. Z. Angew. Math. Mech., 2015:1-15,doi:10.1002/zamm.201500015.
- [34] C.Shen and M.Sun. Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the Aw-Rascle model. J. Differential Equations, 2010, **249**: 3024-3051.
- [35] Smoller J. Shock Waves and Reaction-Diffusion Equation. New York: Springer-Verlag, 1994.
- [36] M.Sun. Interactions of elementary waves for the Aw-Rascle model. SIAM J. Appl. Math., 2009, **69**(6): 1542-1558.
- [37] M.Sun. The exact Riemann solutions to the generalized Chaplygin gas equations with friction. Commun. Nonlinear Sci. Numer. Simulat., 2016, **36**: 342-353.
- [38] W.Sheng, G.Wang and G.Yin. Delta wave and vacuum state for generalized Chaplygin gas dynamics system as pressure vanishes. Nonlinear analysis:real world Appl., 2015, **22**: 115-128.

- [39] W.Sheng and Y.Zeng. Generalized  $\delta$ -entropy condition to Riemann solutions for Chaplygin gas in traffic model. Appl. Math. Mech. Engl.Ed., 2015, **36**(3): 353-364.
- [40] W.Sheng and T.Zhang. The Riemann problem for transportation equations in gas dynamics. Mem Amer Math Soc, 1999, **137**(654).
- [41] D.Tan and T.Zhang. Two-dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws I. Four-J cases, II. Initial data involving some rarefaction waves. J Differential Equations, 1994, **111**: 203-282.
- [42] D.Tan, T.Zhang, Y.Zheng. Delta-shock wave as limits of vanishing viscosity for hyperbolic system of conservation laws. J Differential Equations, 1994, **112**: 1-32.
- [43] H. Tsien. Two dimensional subsonic flow of compressible fluids. J Aeron Sci, 1939, **6**: 399-407.
- [44] G.Wang. The Riemann problem for one dimensional generalized Chaplygin gas dynamics. J Math Anal Appl, 2013, **403**: 403-450.
- [45] Z.Wang, and X.Ding. Uniqueness of generalized solution for the Cauchy problem of transportation equations. Acta Math Scientia, 1997, **17**(3): 341-352.
- [46] Z.Wang, F.Huang and X.Ding. On the Cauchy problem of transportation equations. Acta Math Appl Sinica, 1997, **13**(2): 113-122.
- [47] Z.Wang and Q.Zhang. The Riemann problem with delta initial data for the one-dimensional Chaplygin gas equations, *Acta Mathematica Scientia*, **32B**(3) (2012):pp.825-841.
- [48] Q.Zhang. Cauchy problem for the Aw-Rascle traffic model with Chaplygin pressure, preprint.
- [49] H.Zhang. A non-equilibrium traffic model devoid of gas-like behavior. Transp. Res. Part B, 2002, **36**: 275-290.